

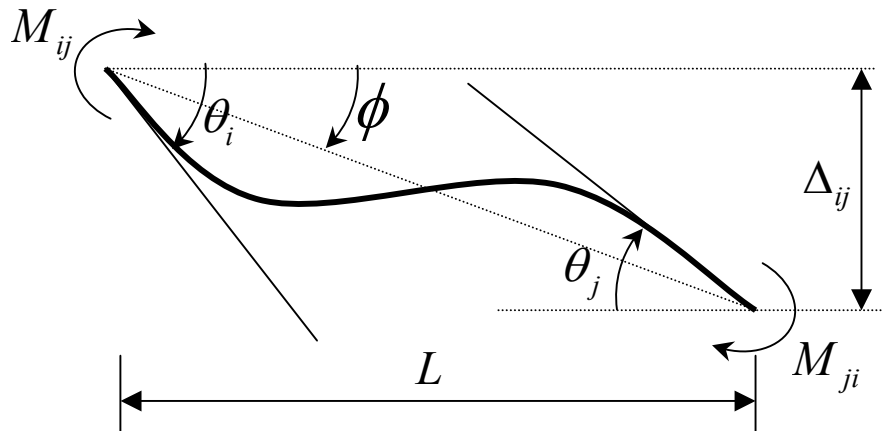
# SLOPE-DEFLECTION METHOD

The slope-deflection method uses displacements as unknowns and is referred to as a displacement method. In the slope-deflection method, the moments at the ends of the members are expressed in terms of displacements and end rotations of these ends.

An important characteristic of the slope-deflection method is that it does not become increasingly complicated to apply as the number of unknowns in the problem increases. In the slope-deflection method the individual equations are relatively easy to construct regardless of the number of unknowns.

## DERIVATION OF THE SLOPE-DEFLECTION EQUATION

When the loads are applied to a frame or to a continuous beam, the member will develop end moments and become deformed as indicated. The notation used in the figure will be followed.

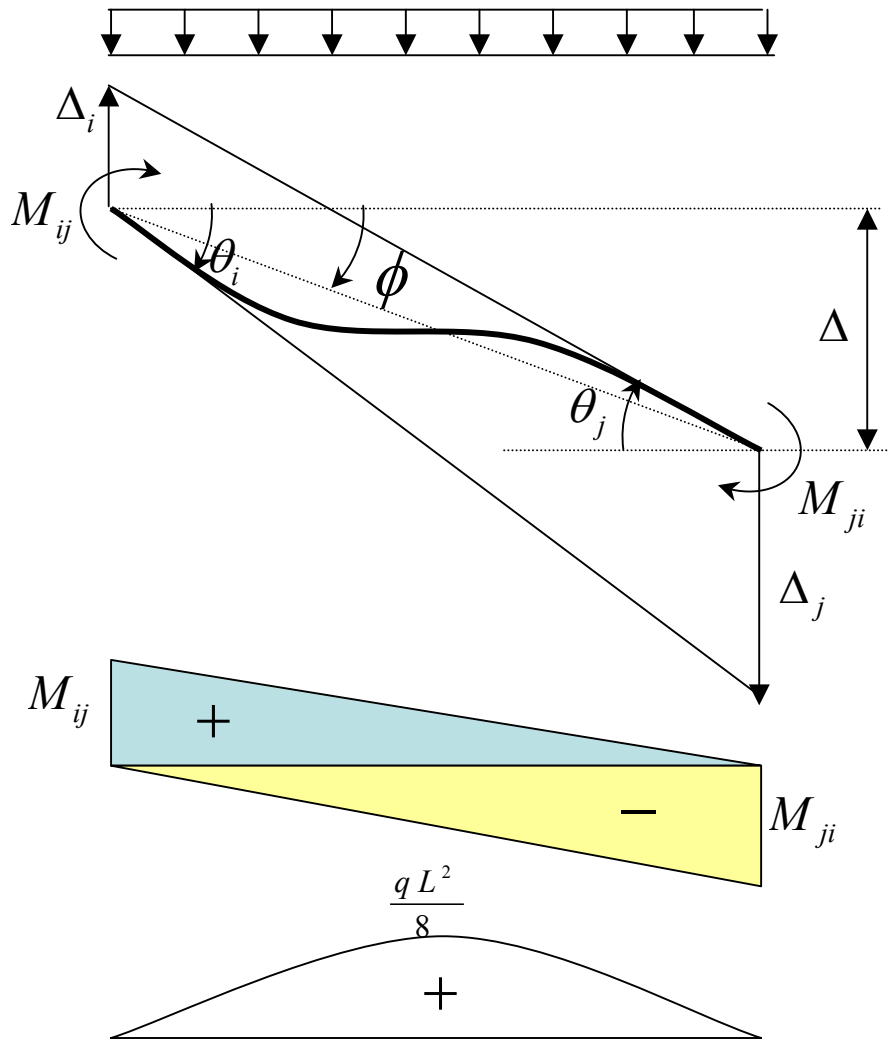


1- The moments at the ends of the member are designated as  $M_{ij}$  and  $M_{ji}$  indicating that they act at ends I and j of member ij.

2- Rotations of ends I and j of the member are denoted by  $\theta_i$  and  $\theta_j$ . Since the rotations of all members of a rigid frame meeting at a common joint are equal, it is customary to refer to each of them as the joint rotation.

3- The term  $\Delta_{ij}$  represents the translation of one end of the member relative to the other end in a direction normal to the axis of the member. Sometimes the rotation of the axis of the member  $\Phi_{ij} = \Delta_{ij}/L$  is used in place of  $\Delta_{ij}$ .

The moments, the rotations at the ends of the member and the rotation of the axis of the member are positive when clockwise.



$$\Delta_j = -\frac{M_{ji}L}{2EI} \frac{L}{3} + \frac{M_{ij}L}{2EI} \frac{2L}{3} + \frac{qL^2}{8EI} \frac{2L}{3} \frac{L}{2}$$

$$\Delta_i = -\frac{M_{ji}L}{2EI} \frac{2L}{3} + \frac{M_{ij}L}{2EI} \frac{L}{3} + \frac{qL^2}{8EI} \frac{2L}{3} \frac{L}{2}$$

$$\theta_i - \phi = \frac{\Delta_j}{L}$$

$$\theta_j - \phi = \frac{\Delta_i}{L}$$

$$\theta_i - \phi = \frac{M_{ij}L}{3EI} - \frac{M_{ji}L}{6EI} + \frac{qL^3}{24EI}$$

$$\theta_j - \phi = -\frac{M_{ij}L}{6EI} + \frac{M_{ji}L}{3EI} - \frac{qL^3}{24EI}$$

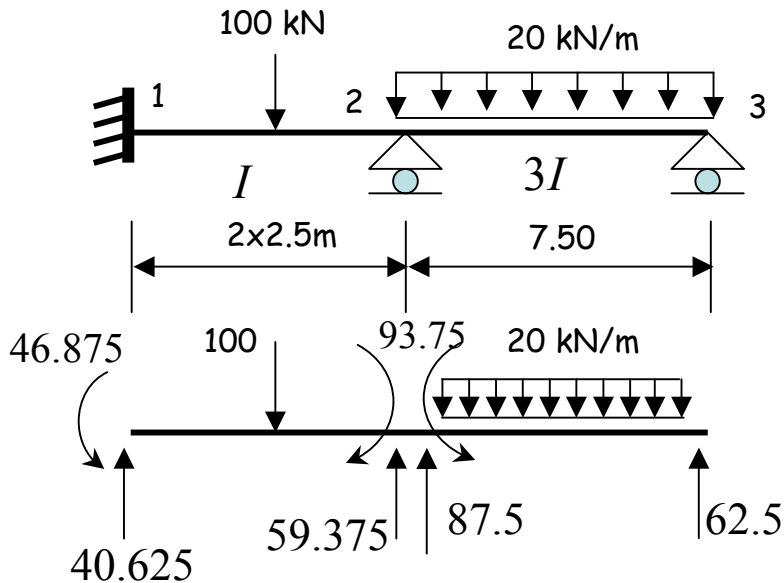
$$M_{ij} = \frac{2EI}{L} (2\theta_i + \theta_j - 3\phi) - \frac{qL^2}{12}$$

$$M_{ji} = \frac{2EI}{L} (\theta_i + 2\theta_j - 3\phi) + \frac{qL^2}{12}$$

Now we wrote  $M_{ij}$  and  $M_{ji}$  in terms of the deformations  $\theta_i$ ,  $\theta_j$ ,  $\phi$  and the external load  $q$  acting on the member. These equations are referred to as **SLOPE-DEFLECTION EQUATIONS**. Slope-deflection equations consider only bending deformations. Deformations due to shear forces and axial forces in bending members are ignored.

$$M_{nf} = \frac{2EI}{L} \left( 2\theta_n + \theta_f - 3 \frac{\Delta}{L} \right) \pm M_{nf}^{FEM}$$

Example: It is required to determine the support moments for the continuous beam.



$$-M_{12}^F = M_{21}^F = \frac{100 \cdot 5}{8} = 62.5 \text{ kNm}$$

$$-M_{23}^F = M_{32}^F = \frac{20 \cdot 7.5^2}{12} = 93.75 \text{ kNm}$$

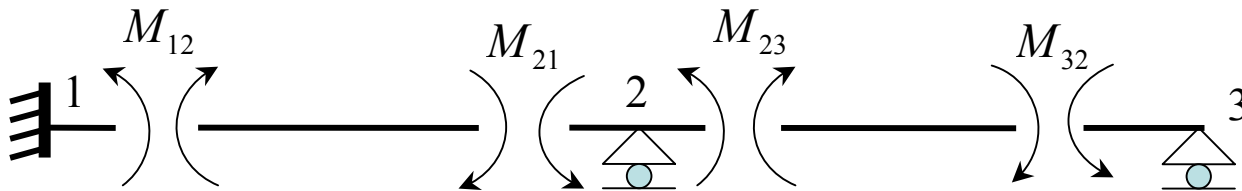
*Slope - Deflection Equations*

$$M_{12} = \frac{2EI}{5} \theta_2 - 62.5 =$$

$$M_{21} = \frac{2EI}{5} 2\theta_2 + 62.5 =$$

$$M_{23} = \frac{6EI}{7.5} (2\theta_2 + \theta_3) - 93.75 =$$

$$M_{32} = \frac{6EI}{7.5} (\theta_2 + 2\theta_3) + 93.75 =$$



*Equilibrium equations of joints*

$$M_{21} + M_{23} = 0$$

$$M_{32} = 0$$

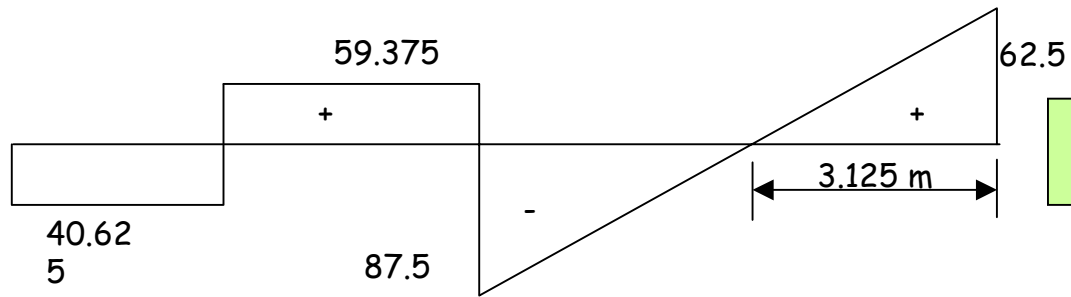
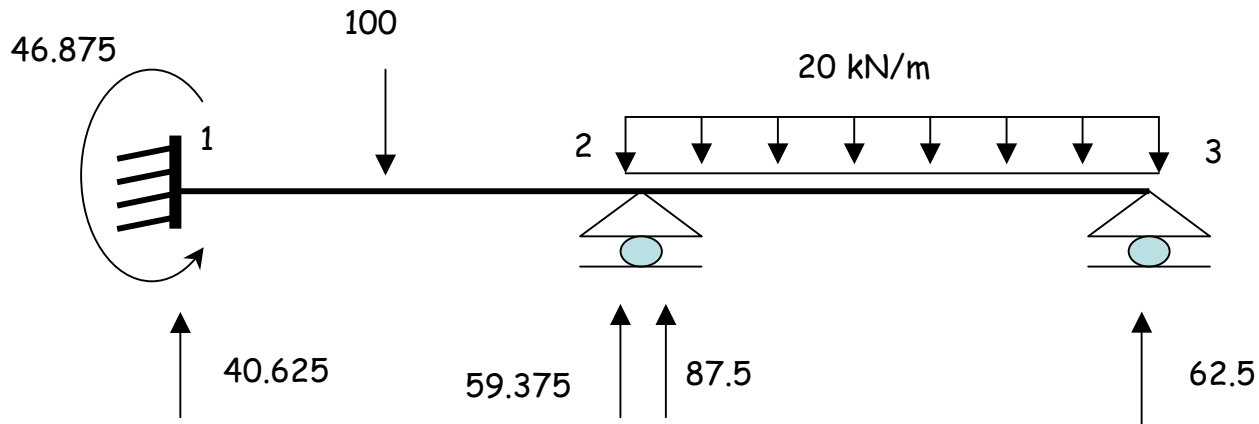
$$2.4EI\theta_2 + 0.8EI\theta_3 = 31.25$$

$$0.8EI\theta_2 + 1.6EI\theta_3 = -93.75 \rightarrow \theta_2 = \frac{39.0625}{EI} \rightarrow \theta_3 = \frac{-78.125}{EI}$$

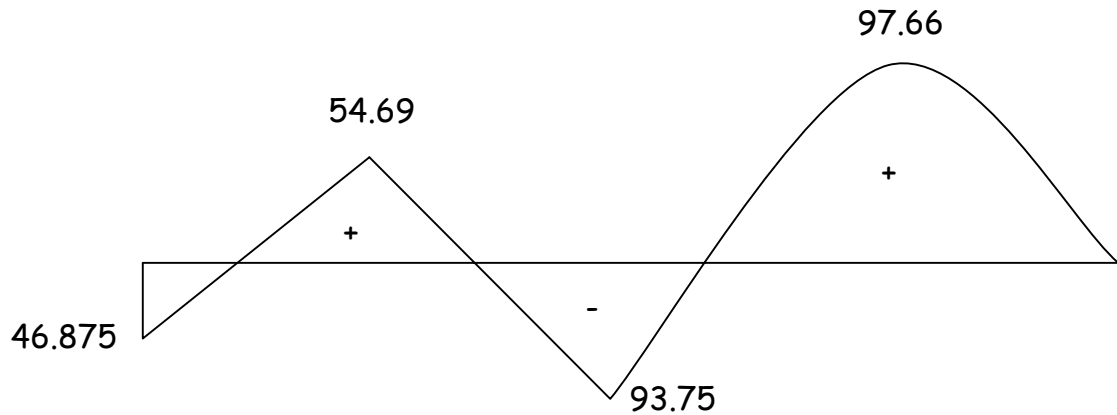
*Substitute these results in slope deflection equations*

$$M_{12} = -46.875 \text{ kNm}, \rightarrow M_{21} = 93.75 \text{ kNm}$$

$$M_{23} = -93.75 \text{ kNm}, \rightarrow M_{32} = 0 \text{ kNm}$$

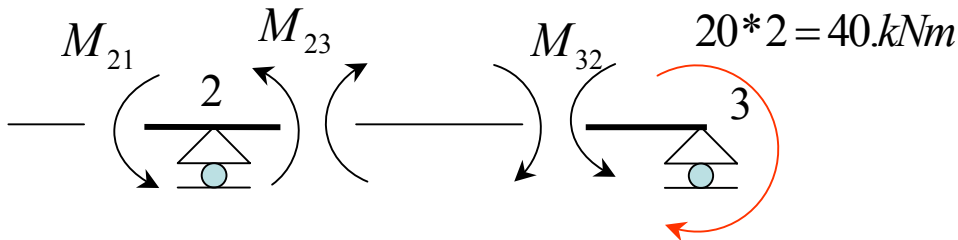
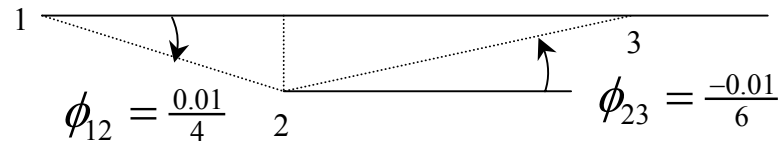
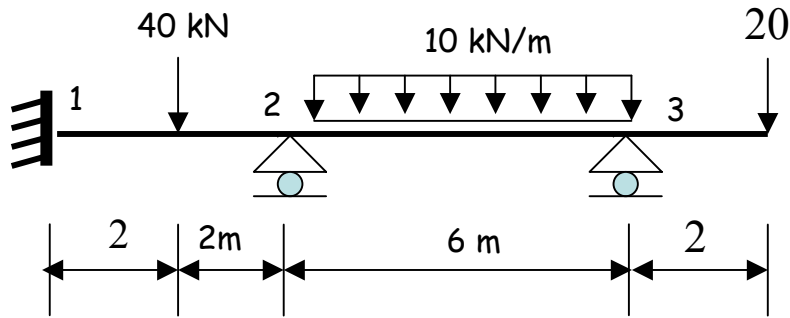


Shear Force Diagram



Bending Moment Diagram

Example: A continuous beam is supported and loaded as shown in the figure. During loading support 2 sinks by 10 mm. Analyze the beam for support moments and reactions.



$$M_{21} + M_{23} = 0$$

$$M_{32} - 40 = 0$$

$$1.667\theta_2 + 0.3333\theta_3 = \frac{51.667}{EI}$$

$$0.3333\theta_2 + 0.6667\theta_3 = -\frac{23.333}{EI}$$

$$\theta_2 = \frac{42.222}{EI} = 2.111 \cdot 10^{-3} \text{ rad}$$

$$\theta_3 = \frac{-56.111}{EI} = -2.8055 \cdot 10^{-3} \text{ rad}$$

$$E = 200 \cdot 10^6 \dots \frac{kN}{m^2}$$

$$I = 100 \cdot 10^{-6} \dots m^4$$

$$EI = 20000 \dots kNm^2$$

$$-M_{12}^F = M_{21}^F = \frac{40 \cdot 4}{8} = 20 kNm$$

$$-M_{23}^F = M_{32}^F = \frac{10 \cdot 6^2}{12} = 30 kNm$$

*Slope - Deflection Equations*

$$M_{12} = \frac{2EI}{4} \left( \theta_2 - 3 \frac{0.01}{4} \right) - 20 =$$

$$M_{21} = \frac{2EI}{4} \left( 2\theta_2 - 3 \frac{0.01}{4} \right) + 20 =$$

$$M_{23} = \frac{2EI}{6} \left( 2\theta_2 + \theta_3 + 3 \frac{0.01}{6} \right) - 30 =$$

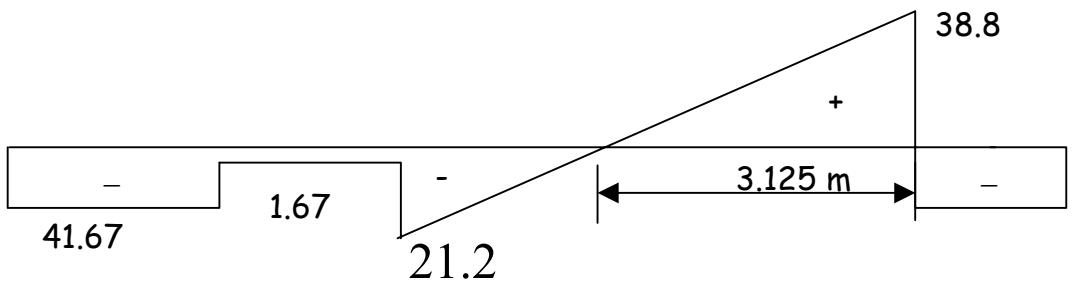
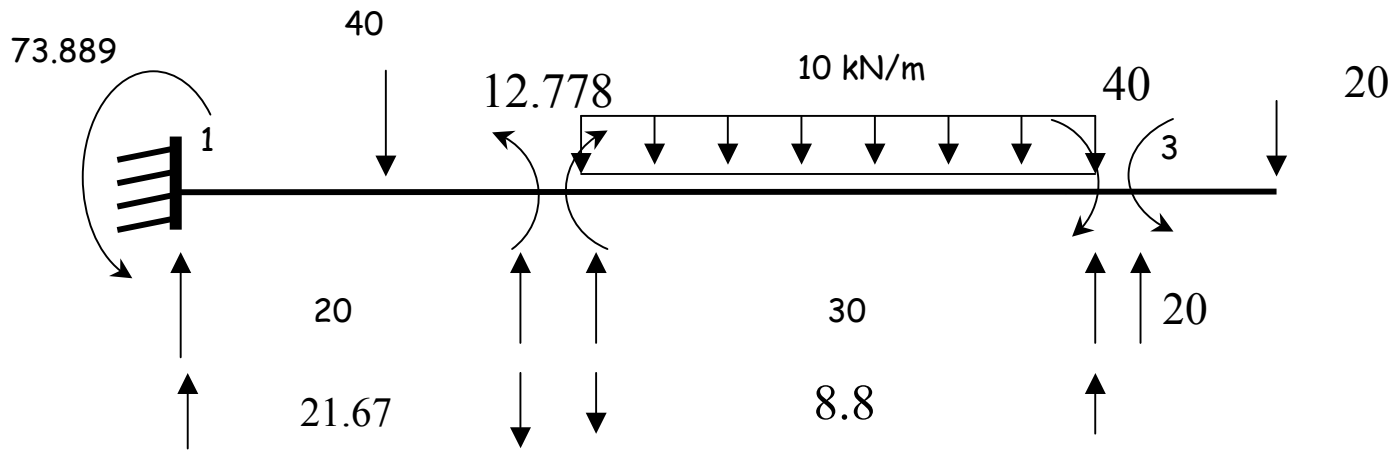
$$M_{32} = \frac{2EI}{6} \left( \theta_2 + 2\theta_3 + 3 \frac{0.01}{6} \right) + 30 =$$

*Substitute these results*

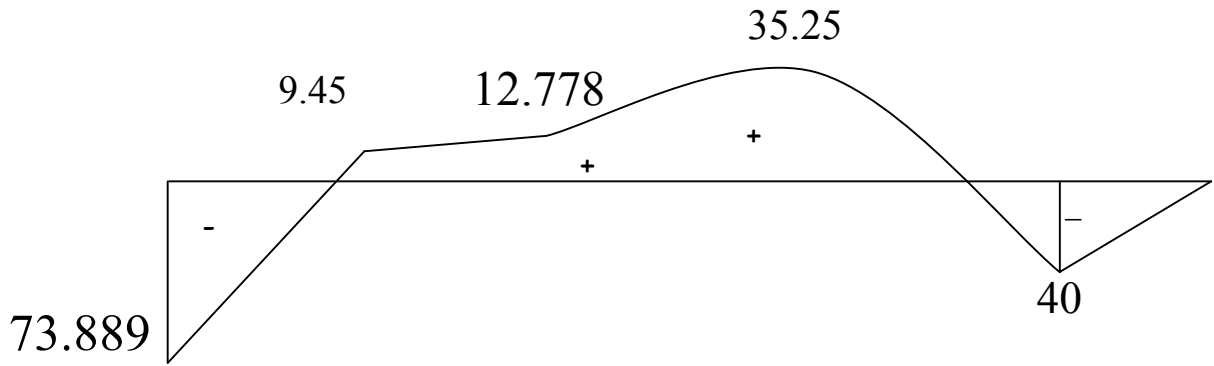
*in slope deflection equations*

$$M_{12} = -73.889 kNm, \rightarrow M_{21} = -12.778 kNm$$

$$M_{23} = 12.778 kNm, \rightarrow M_{32} = 40 kNm$$



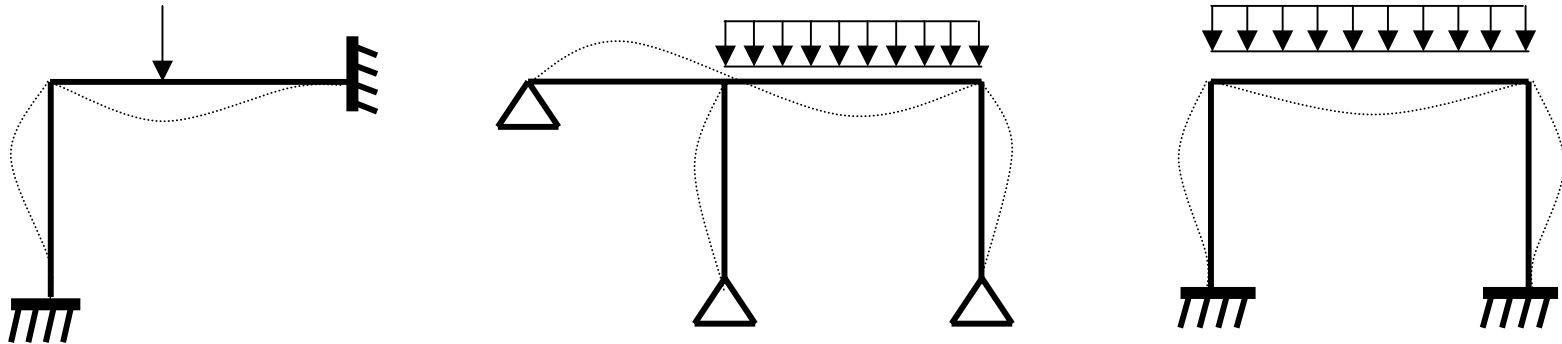
Shear Force Diagram



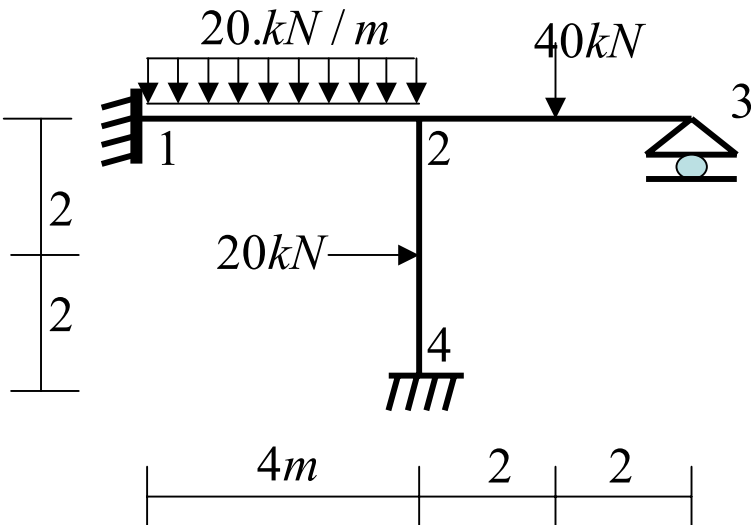
Bending Moment Diagram

## ANALYSIS OF FRAMES WITH NO SIDESWAY

A frame will not side sway, or be displaced to the left or right, provided it is properly restrained. Also no side sway will occur in an unrestrained frame provided it is symmetric with respect to both loading and geometry.



Example: It is required to analyze the frame for moments at the ends of members.  $EI$  is constant for all members.



Fixed-End Moments

$$-M_{12}^F = M_{21}^F = \frac{20 \cdot 4^2}{12} = 26.67 \text{ kNm}$$

$$-M_{23}^F = M_{32}^F = \frac{40 \cdot 4}{8} = 20 \text{ kNm}$$

$$-M_{42}^F = M_{24}^F = \frac{20 \cdot 4}{8} = 10 \text{ kNm}$$

# Slope - Deflection.Equations

$$M_{12} = \frac{2EI}{4}\theta_2 - 26.67 = -27.88$$

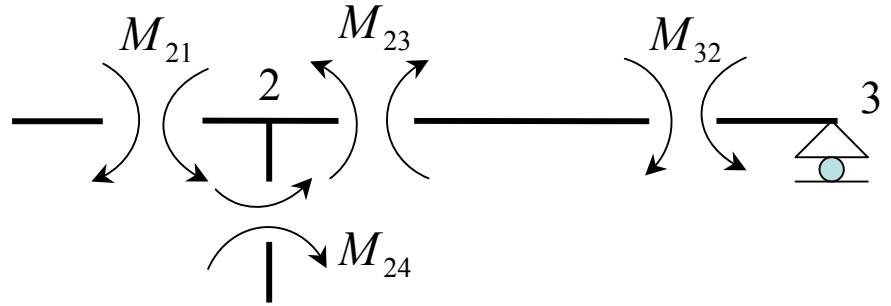
$$M_{21} = \frac{2EI}{4}2\theta_2 + 26.67 = 24.245$$

$$M_{23} = \frac{2EI}{4}(2\theta_2 + \theta_3) - 20 = -31.82$$

$$M_{32} = \frac{2EI}{4}(\theta_2 + 2\theta_3) + 20 = 0$$

$$M_{42} = \frac{2EI}{4}(\theta_2) - 10 = -11.21$$

$$M_{24} = \frac{2EI}{4}(2\theta_2) + 10 = 7.575$$



Equilibrium equations of joints

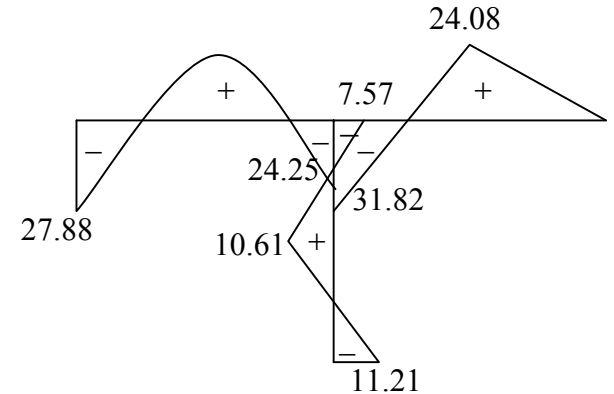
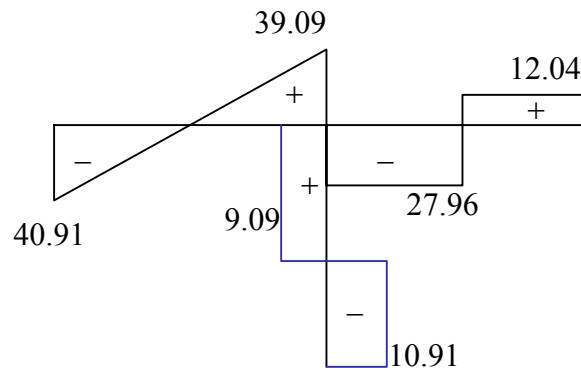
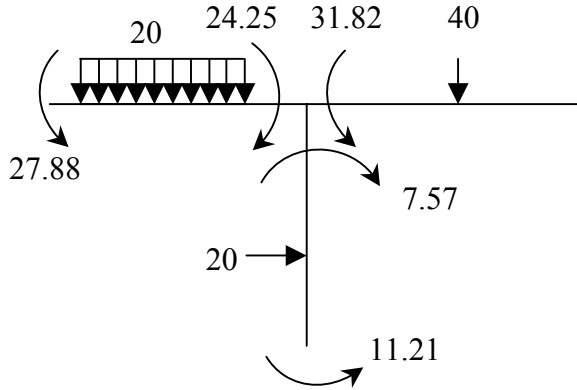
$$M_{21} + M_{23} + M_{24} = 0$$

$$M_{32} = 0$$

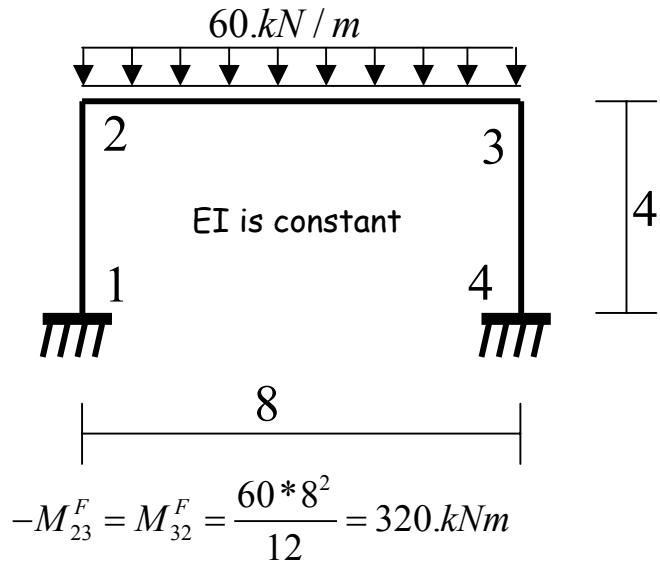
$$\frac{2EI}{4}(6\theta_2 + \theta_3) + 16.67 = 0$$

$$\frac{2EI}{4}(\theta_2 + 2\theta_3) + 20 = 0 \rightarrow \theta_2 = \frac{-2.425}{EI} \rightarrow \theta_3 = \frac{-18.787}{EI}$$

Substitute these results in slope deflection equations



Example: Find Member end moments and draw shear and moment diagrams



Equilibrium equations of joints

$$M_{21} + M_{23} = 0$$

$$M_{32} + M_{34} = 0$$

$$1.5EI\theta_2 + 0.25EI\theta_3 = 320$$

$$0.25EI\theta_2 + 1.5EI\theta_3 = -320 \rightarrow \theta_2 = \frac{256}{EI} \rightarrow \theta_3 = \frac{-256}{EI}$$

Substitute these results in slope deflection equations

$$M_{12} = 128 \text{ kNm}, \rightarrow M_{21} = 256 \text{ kNm}$$

$$M_{23} = -256 \text{ kNm}, \rightarrow M_{32} = 256 \text{ kNm}$$

$$M_{34} = -256 \text{ kNm}, \rightarrow M_{43} = -128 \text{ kNm}$$

Slope - Deflection Equations

$$M_{12} = \frac{2EI}{4}\theta_2 =$$

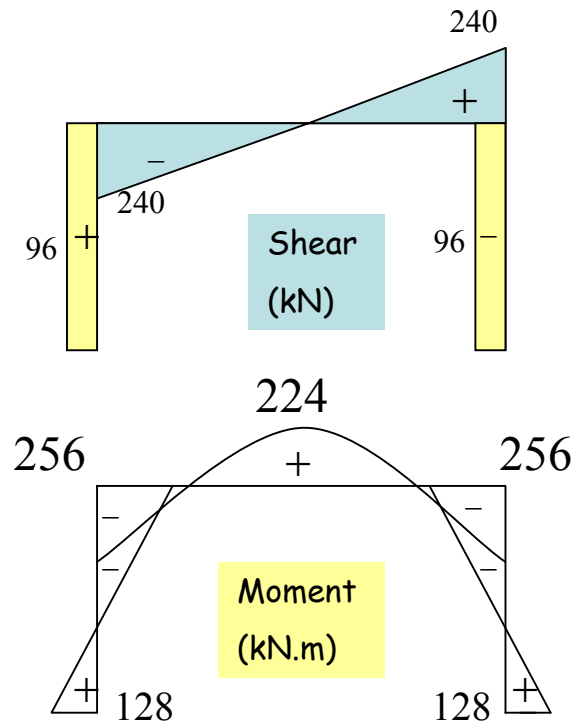
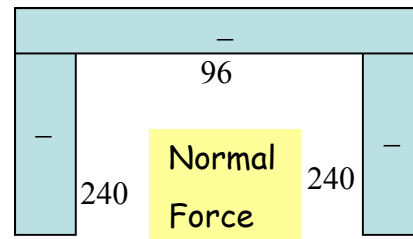
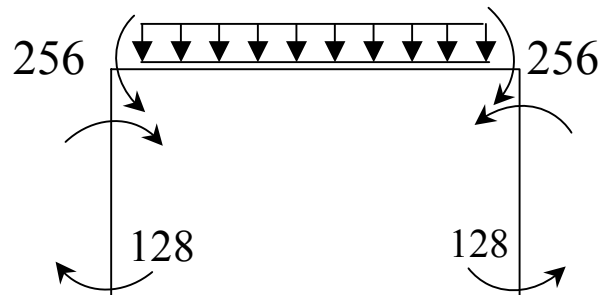
$$M_{21} = \frac{2EI}{4}2\theta_2 =$$

$$M_{23} = \frac{2EI}{8}(2\theta_2 + \theta_3) - 320 =$$

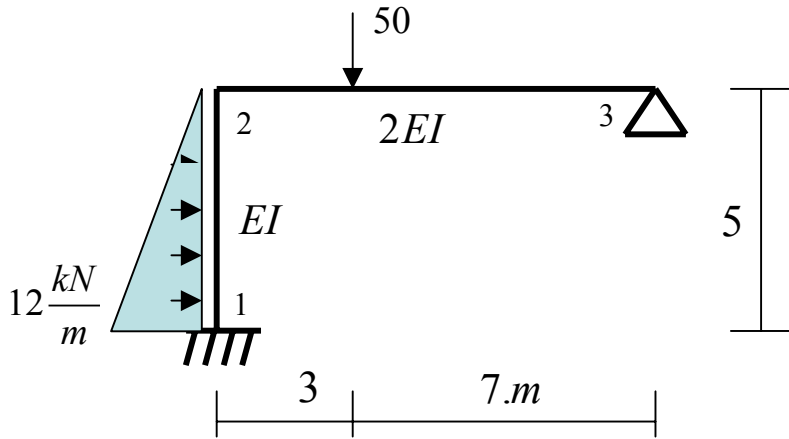
$$M_{32} = \frac{2EI}{8}(\theta_2 + 2\theta_3) + 320 =$$

$$M_{34} = \frac{2EI}{4}(2\theta_3) =$$

$$M_{43} = \frac{2EI}{4}(\theta_3) =$$



Example: Find member end moments and draw the diagrams of the frame



$$-M_{12}^F = \frac{q_0 * L^2}{20} = \frac{12 * 5^2}{20} = 15$$

$$M_{21}^F = \frac{q_0 * L^2}{30} = \frac{12 * 5^2}{30} = 10 \text{ kNm}$$

$$-M_{23}^F = \frac{Pab^2}{L^2} = \frac{50 * 3 * 7^2}{10^2} = 73.5 \text{ kNm}$$

$$M_{32}^F = \frac{Pa^2b}{L^2} = \frac{50 * 3^2 * 7}{10^2} = 31.5 \text{ kNm}$$

*Slope - Deflection Equations*

$$M_{12} = \frac{2EI}{5} \theta_2 - 15 =$$

$$M_{21} = \frac{2EI}{5} 2\theta_2 + 10 =$$

$$M_{23} = \frac{4EI}{10} (2\theta_2 + \theta_3) - 73.5 =$$

$$M_{32} = \frac{4EI}{10} (\theta_2 + 2\theta_3) + 31.5 =$$

$$\theta_2 = \frac{56.61}{EI} \rightarrow \theta_3 = \frac{-67.68}{EI}$$

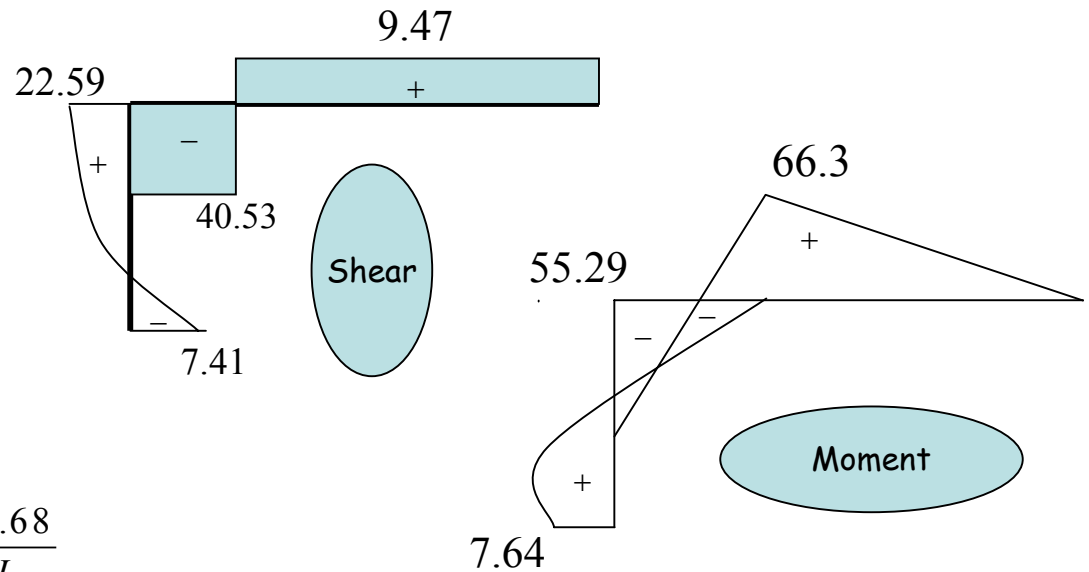
$$M_{21} + M_{23} = 0$$

$$M_{32} = 0$$

*Substitute these results in slope deflection equations*

$$M_{12} = 7.64 \text{ kNm}, \rightarrow M_{21} = 55.29 \text{ kNm}$$

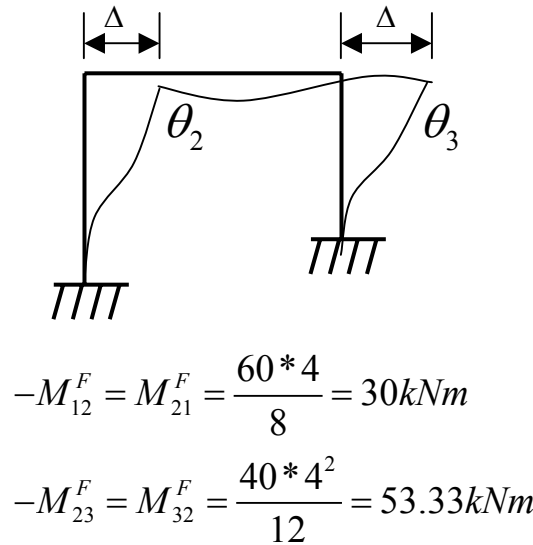
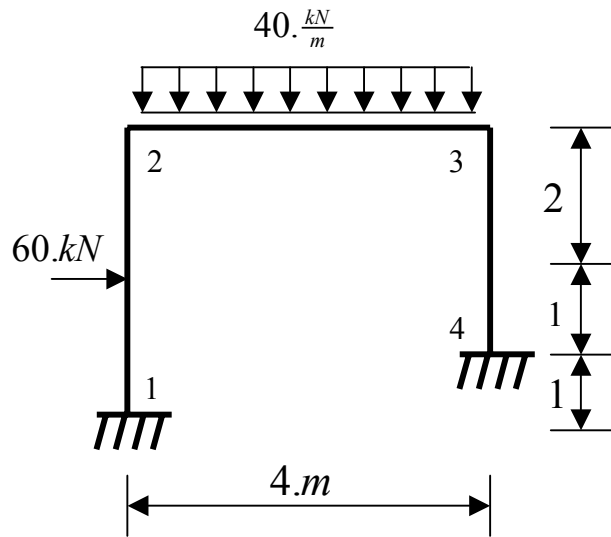
$$M_{23} = -55.29 \text{ kNm}, \rightarrow M_{32} = 0 \text{ kNm}$$



# ANALYSIS OF FRAMES WITH SIDESWAY

A frame will side sway or be displaced to the side when the frame or loading acting on it is non-symmetric. In the analysis of frames with side sway it is necessary to consider the shear forces at the base of the columns and the horizontal external load must be in equilibrium (force equilibrium equation) in addition to the equilibrium of joints.

Example: Using the slope-deflection method determine the end moments of the members and draw the shear force and bending moment diagrams of the frame. EI is constant throughout the frame.



Axial deformation is neglected (no change in length of the members) so the lateral displacement of joint 2 and 3 are equal.

*Slope - Deflection Equations*

$$M_{12} = \frac{2EI}{4} \left( \theta_2 - 3 \frac{\Delta}{4} \right) - 30 =$$

$$M_{21} = \frac{2EI}{4} \left( 2\theta_2 - 3 \frac{\Delta}{4} \right) + 30 =$$

$$M_{23} = \frac{2EI}{4} (2\theta_2 + \theta_3) - 53.33 =$$

$$M_{32} = \frac{2EI}{4} (\theta_2 + 2\theta_3) + 53.33 =$$

$$M_{34} = \frac{2EI}{3} \left( 2\theta_3 - 3 \frac{\Delta}{3} \right) =$$

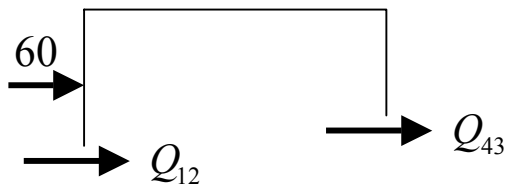
$$M_{43} = \frac{2EI}{3} \left( \theta_3 - 3 \frac{\Delta}{3} \right) =$$

Equilibrium Equations

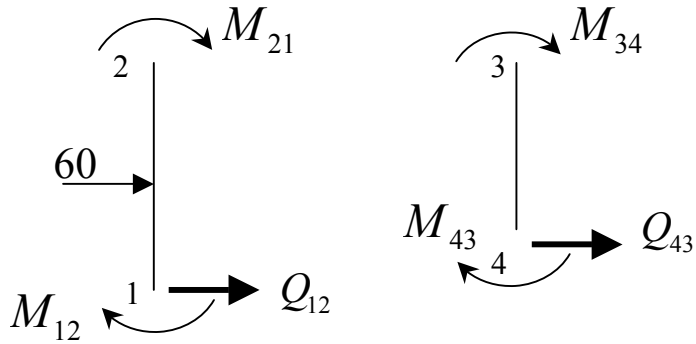
$$M_{21} + M_{23} = 0$$

$$M_{32} + M_{34} = 0$$

$$Q_{12} + Q_{43} + 60 = 0$$



### Shear forces at the base of columns

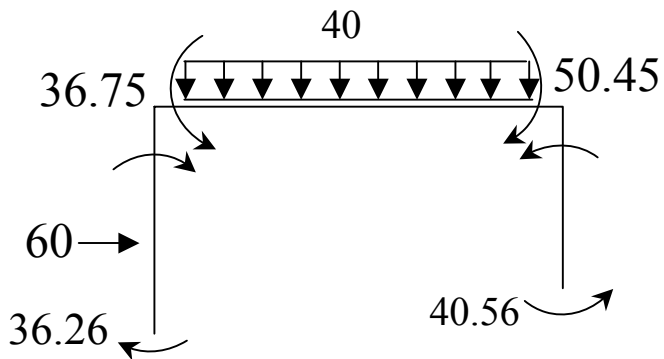


$$\sum M_2 = 0 \rightarrow M_{21} + M_{12} - 4Q_{12} - 2 * 60 = 0$$

$$\sum M_3 = 0 \rightarrow M_{34} + M_{43} - 3Q_{43} = 0$$

$$Q_{12} = \frac{M_{21} + M_{12}}{4} - 30$$

$$Q_{43} = \frac{M_{34} + M_{43}}{3}$$



$$\begin{pmatrix} 3 & 5.333 & -5.06 \\ 4 & 1 & -0.75 \\ 1 & 4.667 & -1.33 \end{pmatrix} \begin{pmatrix} \theta_2 \\ \theta_3 \\ \Delta \end{pmatrix} = \frac{1}{EI} \begin{pmatrix} -240 \\ 46.67 \\ -106.67 \end{pmatrix}$$

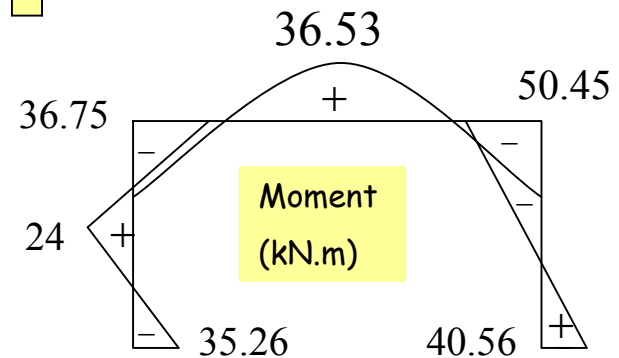
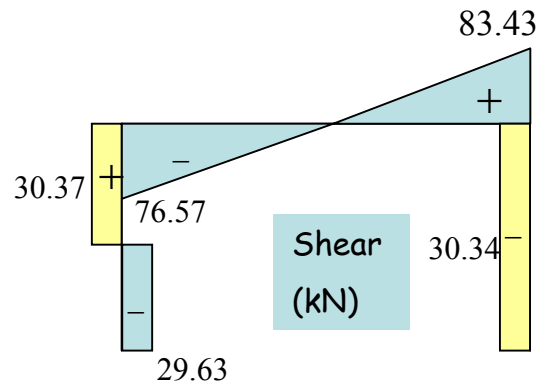
$$\theta_2 = \frac{23.96}{EI} \rightarrow \theta_3 = \frac{-14.857}{EI} \rightarrow \Delta = \frac{45.98}{EI}$$

Substitute these results in slope deflection equations

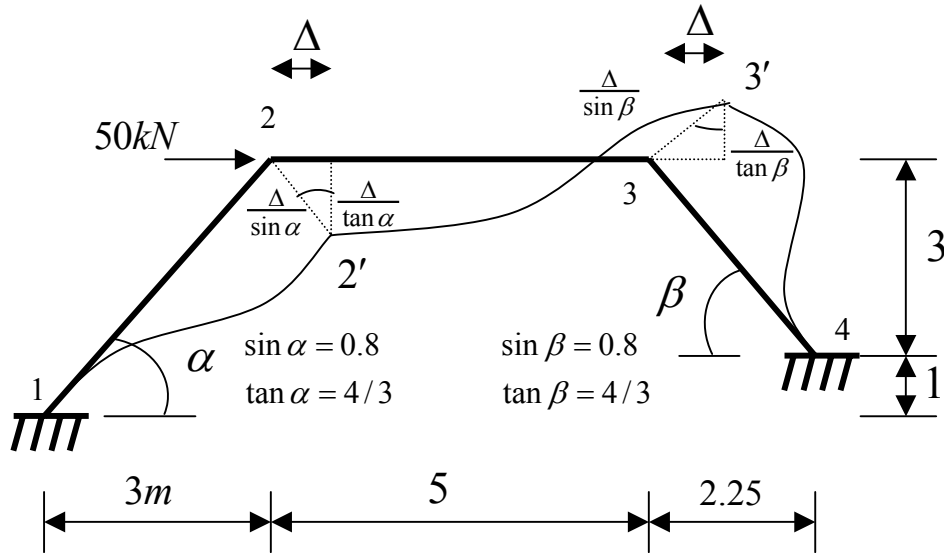
$$M_{12} = -35.26 \text{ kNm}, \rightarrow M_{21} = 36.72 \text{ kNm}$$

$$M_{23} = -36.79 \text{ kNm}, \rightarrow M_{32} = 50.45 \text{ kNm}$$

$$M_{34} = -50.46 \text{ kNm}, \rightarrow M_{43} = -40.56 \text{ kNm}$$



Example: Determine the member end moments of the frame and draw the shear and moment diagrams.



*Slope - Deflection Equations*

$$M_{12} = \frac{2EI}{5} \left( \theta_2 - 3 \frac{\Delta}{5 \sin \alpha} \right) = \frac{2EI}{5} (\theta_2 - 0.75 \Delta) =$$

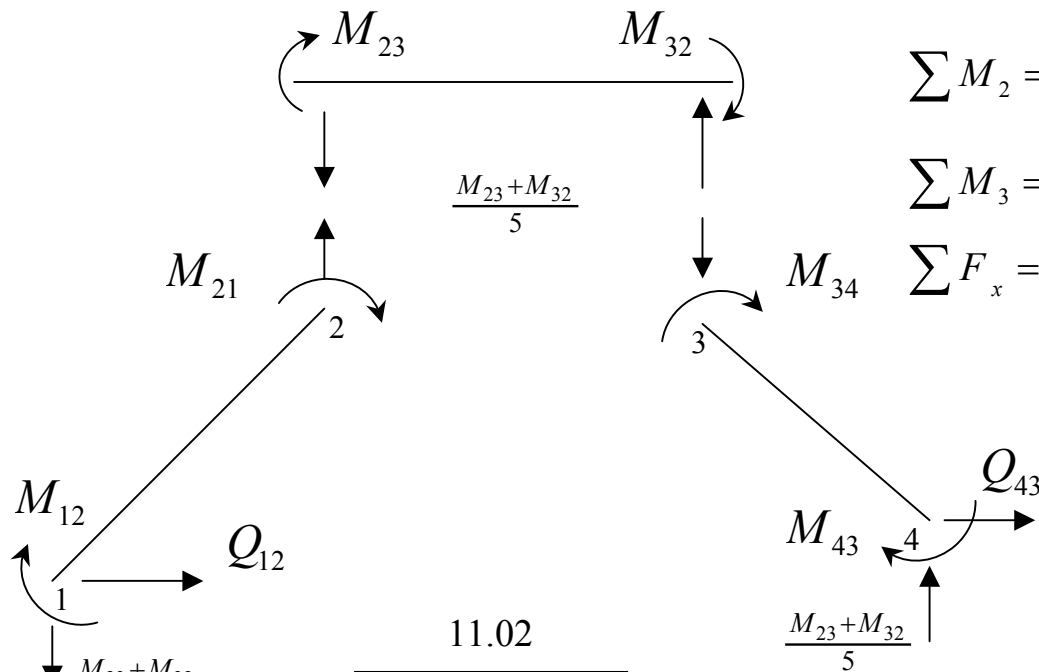
$$M_{21} = \frac{2EI}{5} \left( 2\theta_2 - 3 \frac{\Delta}{5 \sin \alpha} \right) = \frac{2EI}{5} (2\theta_2 - 0.75 \Delta) =$$

$$M_{23} = \frac{2EI}{5} \left( 2\theta_2 + \theta_3 + 3 \frac{\Delta}{5} \left( \frac{1}{\tan \alpha} + \frac{1}{\tan \beta} \right) \right) = \frac{2EI}{5} (2\theta_2 + \theta_3 + 0.9 \Delta) =$$

$$M_{32} = \frac{2EI}{5} \left( \theta_2 + 2\theta_3 + 3 \frac{\Delta}{5} \left( \frac{1}{\tan \alpha} + \frac{1}{\tan \beta} \right) \right) = \frac{2EI}{5} (\theta_2 + 2\theta_3 + 0.9 \Delta) =$$

$$M_{34} = \frac{2EI}{3.75} \left( 2\theta_3 - 3 \frac{\Delta}{3.75 \sin \beta} \right) = \frac{2EI}{3.75} (2\theta_3 - \Delta) =$$

$$M_{43} = \frac{2EI}{3.75} \left( \theta_3 - 3 \frac{\Delta}{3.75 \sin \beta} \right) = \frac{2EI}{3.75} (\theta_3 - \Delta) =$$



$$\sum M_2 = 0 \rightarrow 4Q_{12} + \frac{M_{23} + M_{32}}{5} * 3 - (M_{12} + M_{21}) = 0$$

$$\sum M_3 = 0 \rightarrow 3Q_{43} + \frac{M_{23} + M_{32}}{5} * 2.25 - (M_{43} + M_{34}) = 0$$

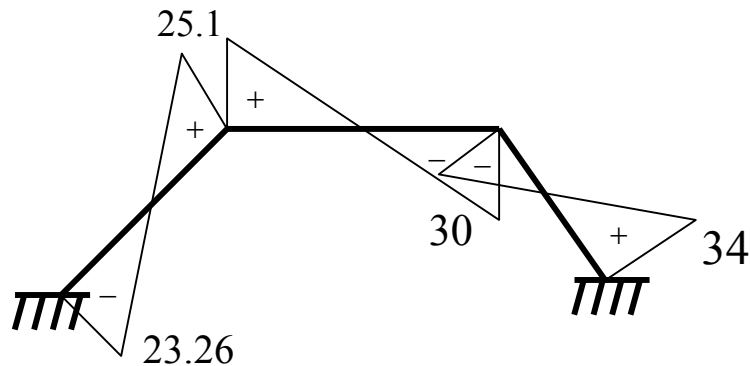
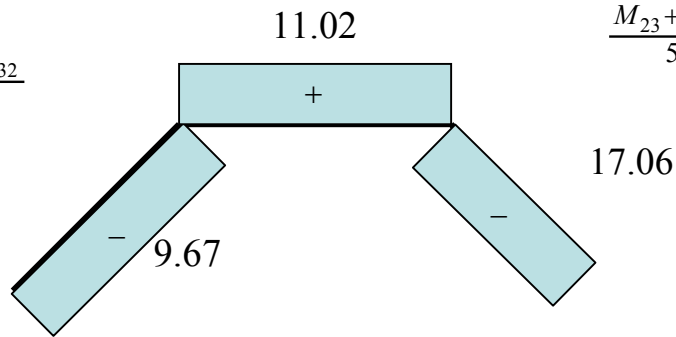
$$\sum F_x = 0 \rightarrow Q_{12} + Q_{43} + 50 = 0$$

Equilibrium Equations

$$M_{21} + M_{23} = 0$$

$$M_{32} + M_{34} = 0$$

$$Q_{12} + Q_{43} + 50 = 0$$



$$\begin{pmatrix} 4 & 1 & 0.15 \\ 0.2 & 0.933 & -0.087 \\ -0.03 & 0.0867 & -0.361 \end{pmatrix} \begin{pmatrix} \theta_2 \\ \theta_3 \\ \Delta \end{pmatrix} = \frac{1}{EI} \begin{pmatrix} 0 \\ 0 \\ -25 \end{pmatrix}$$

$$\theta_2 = \frac{-4.59}{EI} \rightarrow \theta_3 = \frac{7.646}{EI} \rightarrow \Delta = \frac{71.41}{EI}$$

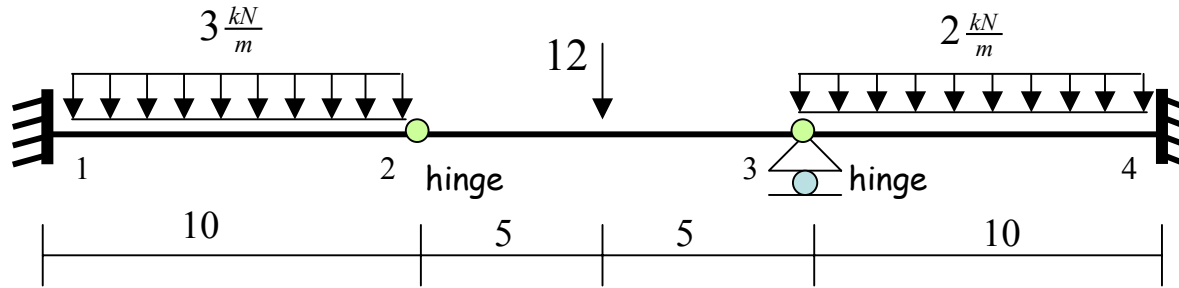
Substitute these results in slope deflection equations

$$M_{12} = -23.26 \text{ kNm}, \rightarrow M_{21} = -25.1 \text{ kNm}$$

$$M_{23} = 25.10 \text{ kNm}, \rightarrow M_{32} = 30 \text{ kNm}$$

$$M_{34} = -30 \text{ kNm}, \rightarrow M_{43} = -34 \text{ kNm}$$

Example: Determine the member end moments and draw the shear and moment diagrams of given continuous beam.



Rotations at the left and right side of the internal hinges are different from each other

*Slope - Deflection Equations*

$$M_{12} = \frac{2EI}{10} \left( \theta_2^L - 3 \frac{\Delta}{10} \right) - 25 =$$

$$M_{21} = \frac{2EI}{10} \left( 2\theta_2^L - 3 \frac{\Delta}{10} \right) + 25 =$$

$$M_{23} = \frac{2EI}{10} \left( 2\theta_2^R + \theta_3^L + 3 \frac{\Delta}{10} \right) - 15 =$$

$$M_{32} = \frac{2EI}{10} \left( \theta_2^R + 2\theta_3^L + 3 \frac{\Delta}{10} \right) + 15 =$$

$$M_{34} = \frac{2EI}{10} (2\theta_3^R) - 16.667 =$$

$$M_{43} = \frac{2EI}{10} (\theta_3^R) + 16.667 =$$

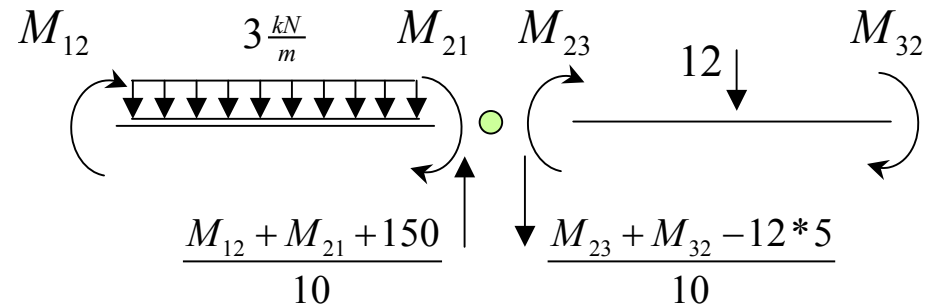
Equilibrium Equations

$$M_{34} = 0 \rightarrow \theta_3^R = 41.667/EI$$

$$M_{21} = 0$$

$$M_{23} = 0$$

$$M_{32} = 0$$



Shear forces at each side of the hinge must be equal to each other

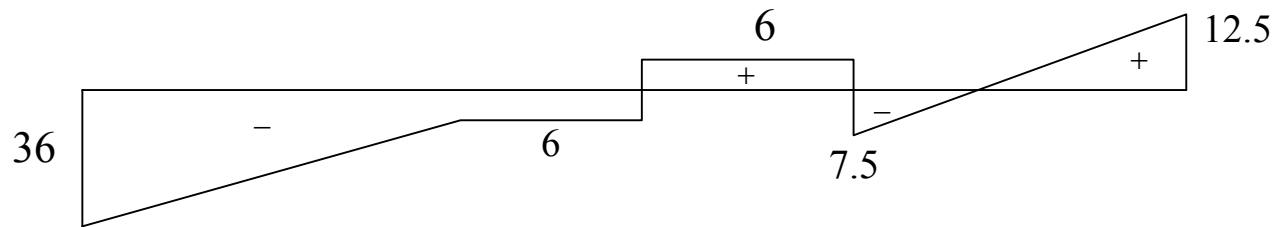
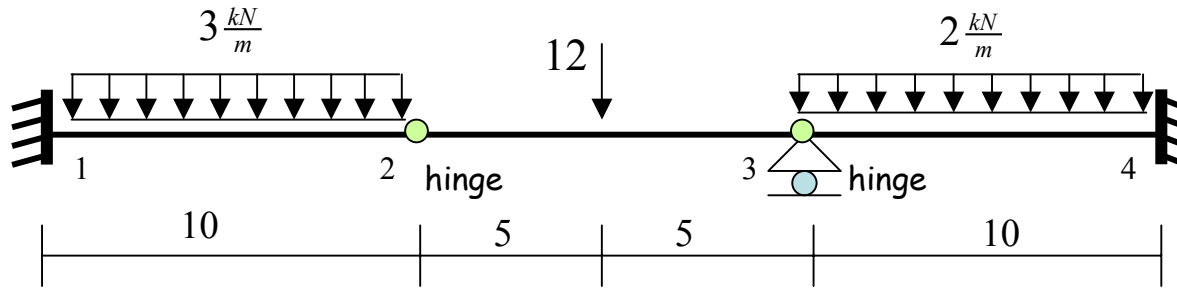
$$\frac{M_{12} + M_{21}}{10} + 15 = \frac{M_{23} + M_{32}}{10} - 6$$

Force Eq. Equation

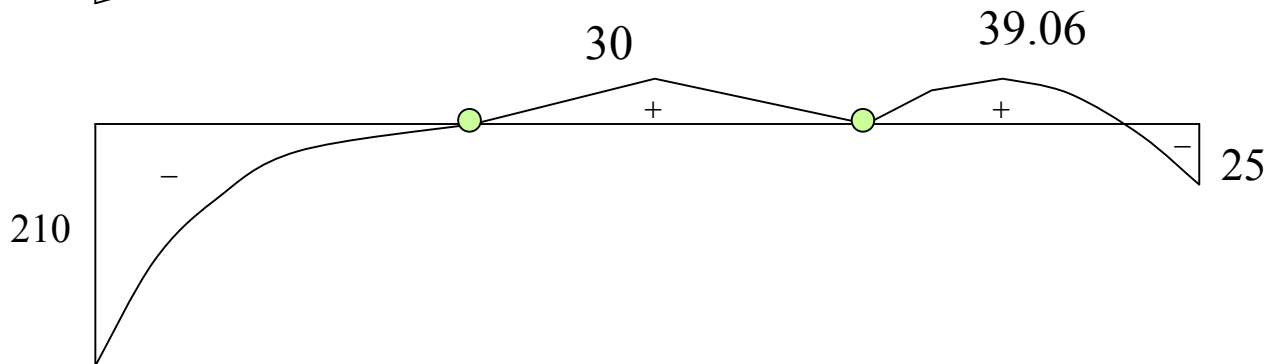
$$\begin{bmatrix} 3 & -3 & -3 & -1.2 \\ 2 & 0 & 0 & -0.3 \\ 0 & 2 & 1 & 0.3 \\ 0 & 1 & 2 & 0.3 \end{bmatrix} \begin{pmatrix} \theta_2^L \\ \theta_2^R \\ \theta_3^L \\ \Delta \end{pmatrix} = \frac{1}{EI} \begin{bmatrix} -1050 \\ -125 \\ 75 \\ -75 \end{bmatrix}$$

$$\theta_2^L = 800/EI \rightarrow \theta_2^R = -500/EI \rightarrow \theta_3^L = -650/EI$$

$$\Delta = 5750/EI$$

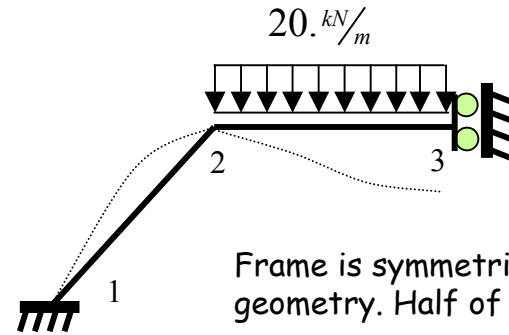
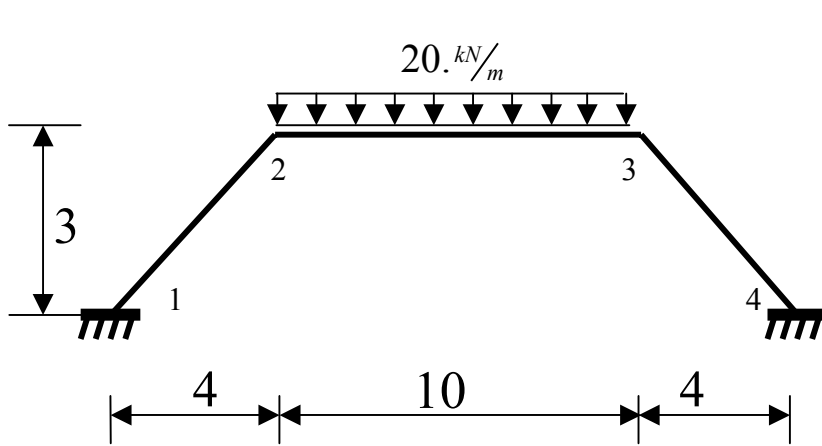


Shear Force Diagram



Bending Moment Diagram

Example: Find the member end moments and draw the shear force and bending moment diagrams of the given frame.



Frame is symmetrical both loading and geometry. Half of the frame can be analyzed

*Slope - Deflection Equations*

$$M_{12} = \frac{2EI}{5} \theta_2 =$$

$$M_{21} = \frac{2EI}{5} 2\theta_2 =$$

$$M_{23} = \frac{2EI}{5} \left( 2\theta_2 - 3 \frac{\Delta}{5} \right) - 41.667 =$$

$$M_{32} = \frac{2EI}{5} \left( \theta_2 - 3 \frac{\Delta}{5} \right) + 41.667 =$$

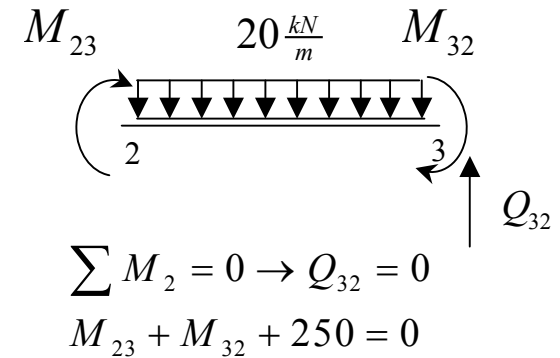
Equilibrium Equations

$$M_{21} + M_{23} = 0$$

$$Q_{32} = 0$$

$$\theta_2 = 166.667/EI$$

$$\Delta = 937.5/EI$$



$$\sum M_2 = 0 \rightarrow Q_{32} = 0$$

$$M_{23} + M_{32} + 250 = 0$$

$$M_{12} = 66.66 \rightarrow M_{21} = 133.33$$

$$M_{23} = -133.33 \rightarrow M_{32} = -116.67$$

